Prove that in any triangle ABC with r, R, s as inradius, circumradius and semiperimeter, respectively, holds inequality

(1) $s \le 2R + (3\sqrt{3} - 4)r$ (Blundon's Inequality).

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Let $x := \frac{r}{s-a} = \tan\frac{A}{2}, y := \frac{r}{s-b} = \tan\frac{B}{2}, z := \frac{r}{s-c} = \tan\frac{C}{2}$.

Then $xy + yz + zx = 1,\ x,y,z > 0$ and $\sum \frac{1}{x} = \sum \frac{s-a}{r} \Leftrightarrow$
$\frac{xy+yz+zx}{xyz} = \frac{s}{r} \Leftrightarrow r = s \cdot xyz$ and $x + y + z = \frac{r}{s-a} + \frac{r}{s-b} + \frac{r}{s-c} =$
$\frac{r((s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a))}{(s-a)(s-b)(s-c)} = \frac{r(ab + bc + ca - s^2)}{sr^2} =$
$\frac{4Rr + r^2}{sr} = \frac{r + 4R}{s} \Leftrightarrow 4R = (x+y+z)s - r.$

Let $p := x + y + z,\ q := xyz.$ Then $r = sq$ and $4R = ps - r = s(p-q)$ and inequality(**1**) becomes $s \le \frac{s(p-q)}{2} + \left(3\sqrt{3} - 4\right)sq \Leftrightarrow$
$2 \le p - q + 6\sqrt{3}\,q - 8q \Leftrightarrow \sqrt{3}\,(2-p) \le 9q\left(2 - \sqrt{3}\right) \Leftrightarrow$

(2) $\left(2\sqrt{3} + 3\right)(2-p) \le 9q.$

Since inequality (**2**) is obvious for $p \ge 2$ then suffices to prove it for $p < 2$.
Due to inequality $(x+y+z)^2 \ge 3(xy+yz+zx)$ and Schure's Inequality $\sum x(x-y)(x-z) \ge 0$ in the form $9xyz \ge 4(x+y+z)(xy+yz+zx) - (x+y+z)^3$ we have $p \ge \sqrt{3}$ and $9q \ge 4p - p^3$, respectively.

Thus, $\sqrt{3} \le p < 2$ and then $9q - \left(2\sqrt{3} + 3\right)(2-p) =$
$(9q - 4p + p^3) + 4p - p^3 - \left(2\sqrt{3} + 3\right)(2-p) =$
$(9q - 4p + p^3) + (2-p)\left(2p + p^2 - \left(2\sqrt{3} + 3\right)\right) =$
$(9q - 4p + p^3) + \left(2 + p + \sqrt{3}\right)(2-p)\left(p - \sqrt{3}\right) \ge 0.$ ■